

Orlicz integrability of additive functionals of Harris ergodic Markov chains

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Abstract

For a Harris ergodic Markov chain $(X_n)_{n \geq 0}$, on a general state space, started from the so called small measure or from the stationary distribution we provide optimal estimates for Orlicz norms of sums $\sum_{i=0}^{\tau} f(X_i)$, where τ is the first regeneration time of the chain. The estimates are expressed in terms of other Orlicz norms of the function f (wrt the stationary distribution) and the regeneration time τ (wrt the small measure). We provide applications to tail estimates for additive functionals of the chain (X_n) generated by unbounded functions as well as to classical limit theorems (CLT, LIL, Berry-Esseen).

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1 Introduction and notation

Consider a Polish space \mathcal{X} with the Borel σ -field \mathcal{B} and let $(X_n)_{n \geq 0}$ be a time homogeneous Markov chain on \mathcal{X} with a transition function $P: \mathcal{X} \times \mathcal{B} \rightarrow [0, 1]$. Throughout the article we will assume that the chain is Harris ergodic, i.e. that there exists a unique probability measure π on $(\mathcal{X}, \mathcal{B})$ such that

$$\|P^n(x, \cdot) - \pi\|_{TV} \rightarrow 0$$

for all $x \in \mathcal{X}$, where $\|\cdot\|_{TV}$ denotes the total variation norm, i.e. $\|\mu\|_{TV} = \sup_{A \in \mathcal{B}} |\mu(A)|$ for any signed measure μ .

One of the best known and most efficient tools of studying such chains is the so called regeneration technique [28, 4], which we briefly recall bellow. We refer the reader to the monographs [29], [26] and [12] for extensive description of this method and restrict ourselves to the basics which we will need to formulate and prove our results.

Below we assume that the chain is Harris ergodic.

One can show that under the above assumptions there exists a set (usually called *small set*) $C \in \mathcal{E}^+ = \{A \in \mathcal{B}: \pi(A) > 0\}$, a positive integer m , $\delta > 0$ and a Borell probability measure ν

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on \mathcal{X} (*small measure*) such that

$$P^m(x, \cdot) \geq \delta \nu(\cdot) \quad (1)$$

for all $x \in C$. Moreover one can always choose m and ν in such a way that $\nu(C) > 0$.

Existence of the above objects allows for redefining the chain (possibly on an enlarged probability space) together with auxiliary regeneration structure. More precisely, one defines the sequence $(\tilde{X}_n)_{n \geq 0}$ and a sequence $(Y_n)_{n \geq 0}$ by requiring that \tilde{X}_0 have the same distribution as X_0 and specifying the conditional probabilities

$$\begin{aligned} & \mathbb{P}(Y_k = 1, \tilde{X}_{km+1} \in dx_1, \dots, \tilde{X}_{(k+1)m-1} \in dx_{m-1}, \tilde{X}_{(k+1)m} \in dy | \mathcal{F}_{km}^{\tilde{X}}, \mathcal{F}_{k-1}^Y, \tilde{X}_{km} = x) \\ &= P(Y_k = 1, \tilde{X}_{km+1} \in dx_1, \dots, \tilde{X}_{(k+1)m-1} \in dx_{m-1}, \tilde{X}_{(k+1)m} \in dy | \tilde{X}_0 = x) \\ &= \mathbf{1}_{\{x \in C\}} \frac{\delta \nu(dy)}{P^m(x, dy)} P(x, dx_1) \cdots P(x_{m-1}, dy), \end{aligned}$$

where $\mathcal{F}_{km}^{\tilde{X}} = \sigma((\tilde{X}_i)_{i \leq km})$ and $\mathcal{F}_{k-1}^Y = \sigma((Y_i)_{i \leq k-1})$.

One can easily check that (\tilde{X}_n) has the same distribution as (X_n) and so we may and will identify the two sequences (we will suppress the tilde). The auxiliary variables Y_n can be used to introduce some independence which allows to recover many results for Markov chains from corresponding statements for the independent (or one-dependent) case. Indeed, observe that if we define the stopping times

$$\tau(0) = \inf\{k \geq 0, Y_k = 1\}, \quad \tau(i) = \inf\{k > \tau(i-1) : Y_k = 1\}, \quad i = 1, 2, \dots,$$

then the blocks $R_0 = (X_0, \dots, X_{\tau(0)m+m-1})$, $R_i = (X_{m(\tau(i-1)+1)}, \dots, X_{m\tau(i)+m-1})$ are one-dependent, i.e. for all k $\sigma(R_i, i < k)$ is independent of $\sigma(R_i, i > k)$. In the special case, when $m = 1$ (the so called *strongly aperiodic case*) the blocks R_i are independent. Moreover, for $i \geq 1$ the blocks R_i form a stationary sequence.

In particular for any function $f: \mathcal{X} \rightarrow \mathbb{R}$, the corresponding additive functional $\sum_{i=1}^n f(X_i)$ can be split (modulo the initial and final segment) into a sum (of random length) of one-dependent (independent for $m = 1$) identically distributed summands

$$s_i(f) = \sum_{j=m(\tau(i)+1)}^{m\tau(i+1)+m-1} f(X_j).$$

A crucial and very useful fact is the following equality, which follows from Pitman's occupation measure formula ([33, 34], see also Theorem 10.0.1 in [26]).

$$\mathbb{E}_\nu \sum_{i=0}^{\tau(0)} F(X_{mi}, Y_i) = \delta^{-1} \pi(C)^{-1} \mathbb{E}_\pi F(X_0, Y_0), \quad (2)$$

where by \mathbb{E}_μ we denote the expectation for the process with X_0 distributed according to the measure μ .

It is also worth noting that the distribution of $s_i(f)$ is equal to the distribution of

$$S = S(f) = \sum_{i=0}^{\tau(0)m+m-1} f(X_i)$$

provided that X_0 is distributed according to ν .

In particular, by (2) this easily implies that

$$\mathbb{E}s_i(f) = \delta^{-1}\pi(C)^{-1}m \int_{\mathcal{X}} f d\pi. \quad (3)$$

The above technique of decomposing additive functionals of Markov chains into independent or almost independent summands has proven to be very useful in studying limit theorems for Markov chains (see e.g. [29, 26, 12, 8, 9, 20, 37]) as well as in obtaining non-asymptotic concentration inequalities (see e.g. [13, 14, 1, 2]). The basic difficulty of this approach is providing proper integrability for the variable S . This is usually achieved either via pointwise drift conditions (e.g. [26, 5, 14, 2]), especially important in Markov Chain Monte Carlo algorithms or other statistical applications, when not much information regarding the behaviour of f with respect to the stationary measure is available. Such drift conditions are also useful for quantifying the ergodicity of the chain, measured in terms of integrability of the regeneration time $T = \tau(1) - \tau(0)$ (which via coupling constructions can be translated in the language of total variation norms or mixing coefficients).

Another line of research is more theoretic and concerns the behaviour of the stationary chain. It is then natural to impose conditions concerning integrability of f with respect to the measure π and to assume some order of ergodicity of the chain.

Classical assumptions about integrability of T are of the form $\mathbb{E}T^\alpha < \infty$ or $\mathbb{E}\exp(\theta T) < \infty$, which corresponds to polynomial or geometric ergodicity of the chain. However recently new modified drift conditions have been introduced [15, 14], which give other orders of integrability of T corresponding to various subgeometric rates of ergodicity. Chains satisfying such drift conditions appear naturally in Markov Chain Monte Carlo algorithms or analysis of nonlinear autoregressive models [15].

From this point of view it is natural to ask questions concerning more general notions of integrability of the variable S . In this note we will focus on Orlicz integrability. Recall that $\varphi: [0, \infty) \rightarrow \mathbb{R}_+$ is called a Young functions if it is strictly increasing, convex and $\varphi(0) = 0$. For a real random variable X we define the Orlicz norm corresponding to φ as

$$\|X\|_\varphi = \inf\{C > 0: \mathbb{E}\varphi(|X|/C) \leq 1\}.$$

The Orlicz space associated to φ is the set L_φ of random variables X such that $\|X\|_\varphi < \infty$.

In what follows, we will deal with various underlying measures on the state space \mathcal{X} or on the space of trajectories of the chain. To stress the dependence of the Orlicz norm on the initial distribution μ of the chain (X_n) we will denote it by $\|\cdot\|_{\mu,\varphi}$, e.g. $\|S\|_{\pi,\varphi}$ will denote the φ -Orlicz norm of the functional S for the stationary chain, whereas $\|S\|_{\nu,\varphi}$ the φ -Orlicz norm of the same functional for the chain started from initial distribution ν . We will also denote by $\|f\|_{\mu,\rho}$ the ρ -Orlicz norm of the function $f: \mathcal{X} \rightarrow \mathbb{R}$ when the underlying probability measure is

μ . Although the notation is the same for Orlicz norms of functionals of the Markov chains and functions on \mathcal{X} , the meaning will always be clear from the context and thus should not lead to misunderstanding.

Remarks 1. Note that the distribution of T is independent of the initial distribution of the chain and is equal to the distribution of $\tau(0) + 1$ for the chain starting from the measure ν . Thus $\|T\|_\psi = \|\tau(0) + 1\|_{\nu, \psi}$.

2. In [31], the authors consider ergodicity of order ψ of a Markov chain, for a special class of nondecreasing functions $\psi: \mathbb{N} \rightarrow \mathbb{R}_+$. They call a Markov chain ergodic of order ψ iff $\mathbb{E}_\nu \psi^\circ(T) < \infty$, where $\psi^\circ(n) = \sum_{i=1}^n \psi(i)$. Since ψ° can be extended to a convex increasing function, one can easily see that this notion is closely related to the finiteness of a proper Orlicz norm of T (related to properly shifted function ψ°).

We will be interested in the following two closely related questions

Question 1 Given two Young functions φ and ψ and a Markov chain (X_n) such that $\|T\|_\psi < \infty$, what do we have to assume about $f: \mathcal{X} \rightarrow \mathbb{R}$ to guarantee that $\|S\|_{\nu, \varphi} < \infty$ (resp. $\|S\|_{\pi, \varphi} < \infty$)?

Question 2 Given two Young functions ρ and ψ , a Markov chain (X_n) such that $\|T\|_\psi < \infty$ and $f: \mathcal{X} \rightarrow \mathbb{R}$, such that $\|f\|_{\pi, \rho} < \infty$, what can we say about the integrability of S for the chain started from ν or from π ?

As it turns out, the answers to both questions are surprisingly explicit and elementary. We present them in Section 2 (Theorems 2, 9, Corollaries 6, 14). The upper estimates have very short proofs, which rely only on elementary properties of Orlicz functions and the formula (3). They are also optimal as can be seen from Propositions 3, 10 and Theorem 4 proven in Section 3 by constructing a general class of examples.

We would like to stress that despite being elementary, both the estimates and the counterexamples have non-trivial applications (some of which we present in the last section) and therefore are of considerable interest. For example when specialized to $\varphi(x) = x^2$, the estimates give optimal conditions for the CLT or LIL for Markov chains under assumptions concerning the rate of ergodicity and integrability of the test functions in the stationary case.

In the following sections of the article we present the estimates, demonstrate their optimality and provide applications to limit theorems and tail estimates. For the reader's convenience we gather all the basic facts about Orlicz spaces which are used in the course of the proof in the appendix (we refer the reader to the monographs [21, 24, 35] for more detailed account on this class of Banach spaces).

2 Main estimates

To simplify the notation in what follows we will write τ instead of $\tau(0)$.

2.1 The chain started from ν

Assumption (A) We will assume that

$$\lim_{x \rightarrow 0} \psi(x)/x = 0 \text{ and } \psi(1) \geq 1.$$

Since any Young function on a probability space is equivalent to a function satisfying this condition (see the definition of domination and equivalence of functions below) it will not decrease the generality of our estimates while allowing to describe them in a more concise manner. In particular it assures the correctness of the following definition (where by a generalized Young function we mean a nondecreasing convex function $\rho: [0, \infty) \rightarrow [0, \infty]$ with $\rho(0) = 0$, $\lim_{x \rightarrow \infty} \rho(x) = \infty$).

Definition 1. Let φ and ψ be Young functions. Assume that ψ satisfies the assumption (A). Define the generalized Young function $\rho = \rho_{\varphi, \psi}$ by the formula

$$\rho(x) = \sup_{y \geq 0} \frac{\varphi(xy) - \psi(y)}{y}.$$

Theorem 2. Let φ and ψ be Young functions. Assume that ψ satisfies the assumption (A). Let $\rho = \rho_{\varphi, \psi}$. Then for any Harris ergodic Markov chain (X_n) , a small set C and a measure ν satisfying (1), we have

$$\left\| \sum_{j=0}^{m\tau+m-1} f(X_j) \right\|_{\nu, \varphi} \leq 2m\|\tau + 1\|_{\nu, \psi} \|f\|_{\pi, \rho}. \quad (4)$$

Proof. Let $a = \|\tau + 1\|_{\nu, \psi}$, $b = \|f\|_{\pi, \rho}$. We have

$$\begin{aligned} \mathbb{E}_{\nu} \varphi\left(\frac{S}{abm}\right) &= \mathbb{E}_{\nu} \varphi\left(\frac{\sum_{j=0}^{m\tau+m-1} f(X_j)}{abm}\right) \\ &\leq \mathbb{E}_{\nu} \sum_{j=0}^{m\tau+m-1} \frac{\varphi(f(X_j)b^{-1}(\tau+1)a^{-1})}{(\tau+1)m} \\ &\leq \mathbb{E}_{\nu} \sum_{j=0}^{m\tau+m-1} \frac{\rho(f(X_j)b^{-1})}{am} + \mathbb{E}_{\nu} \sum_{j=0}^{m\tau+m-1} \frac{\psi((\tau+1)a^{-1})}{(\tau+1)m} \\ &= \delta^{-1}\pi(C)^{-1}a^{-1}\mathbb{E}_{\pi}\rho(f(X_0)b^{-1}) + \mathbb{E}_{\nu}\psi((\tau+1)a^{-1}), \end{aligned}$$

where the first inequality follows from Jensen's inequality, the second one from the definition of the function ρ and the last equality from (3). Let us now notice that another application of (3) gives

$$\mathbb{E}_{\nu}(\tau + 1) = \delta^{-1}\pi(C)^{-1}.$$

Thanks to the assumption $\psi(1) \geq 1$, we have $\mathbb{E}_{\nu}\psi((\tau+1)\delta\pi(C)) \geq \psi(\mathbb{E}_{\nu}(\tau+1)\delta\pi(C)) = \psi(1) \geq 1$, which implies that $a \geq \delta^{-1}\pi(C)^{-1}$. Combined with the definition of a and b this gives

$$\mathbb{E}_{\nu} \varphi\left(\frac{S}{abm}\right) \leq 2$$

and hence $\mathbb{E}_{\nu} \varphi(S/(2abm)) \leq \mathbb{E}_{\nu} 2^{-1} \varphi(S/abm) \leq 1$, which ends the proof. \square

As one can see the proof is very simple. At the same time, it turns out that the estimate given in Theorem 2 is optimal (up to constants) and thus answers completely Question 1 for the chain starting from ν . Below we present two results on optimality of Theorem 2 whose proofs are postponed to the next section.

Domination and equivalence of functions Consider two functions $\rho_1, \rho_2: [0, \infty) \rightarrow [0, \infty]$. As is classical in the theory of Orlicz spaces with respect to probabilistic measures, we say that ρ_2 dominates ρ_1 (denoted by $\rho_1 \preceq \rho_2$) if there exist positive constants C_1, C_2 and x_0 , such that

$$\rho_1(x) \leq C_1 \rho_2(C_2 x) \quad (5)$$

for $x \geq x_0$. One can easily check that if ρ_i are Young functions then $\rho_1 \preceq \rho_2$ iff there is an inclusion and comparison of norms between the corresponding Orlicz spaces. We will say that ρ_1 and ρ_2 are equivalent ($\rho_1 \simeq \rho_2$) iff $\rho_1 \preceq \rho_2$ and $\rho_2 \preceq \rho_1$. One can also easily check that two Young functions are equivalent iff they define equivalent Orlicz norms (and the same remains true for functions equivalent to Young functions). Note also that if (5) holds and ρ_2 is a Young function then $\rho_1(x) \leq \rho_2(\max(C_1, 1)C_2 x)$.

Our first optimality result is

Proposition 3 (Weak optimality of Theorem 2). *Let φ and ψ be as in Theorem 2. Assume that a Young function ρ has the property that for every Harris ergodic chain (X_n) , a small set C , a small measure ν with $\|\tau\|_{\nu, \psi} < \infty$ and every function $f: \mathcal{X} \rightarrow \mathbb{R}$ such that $\|f\|_{\pi, \rho} < \infty$, we have $\|S(f)\|_{\nu, \varphi} < \infty$. Then $\rho_{\varphi, \psi} \preceq \rho$.*

It turns out that if we assume something more about the functions φ and ψ , the above proposition can be considerably strengthened.

Theorem 4 (Strong optimality of Theorem 2). *Let φ, ψ and ρ be as in Theorem 2. Assume additionally that $\varphi^{-1} \circ \psi$ is equivalent to a Young function. Let Y be a random variable such that $\|Y\|_{\rho} = \infty$. Then there exists a Harris ergodic Markov chain (X_n) on some Polish space \mathcal{X} , with stationary distribution π , a small set C , a small measure ν and a function $f: \mathcal{X} \rightarrow \mathbb{R}$, such that the distribution of f under π is equal to the law of Y , $\|\tau\|_{\nu, \psi} < \infty$ and $\|S(f)\|_{\nu, \varphi} = \infty$.*

Remarks 1. In the last section we will see that the above theorem for $\varphi(x) = x^2$ can be used to construct examples of chains violating the central limit theorem.

2. We do not know if the additional assumption on convexity of $\varphi^{-1} \circ \psi$ is needed in the above Theorem.

3. In fact in the construction we provide the set C is an atom for the chain (i.e. in the minorization condition $m = 1$ and $\delta = 1$).

The above results give a fairly complete answer to Question 1 for a chain started from a small measure. We will now show that Theorem 2 can be also used to derive the answer to Question 2.

Recall that the Legendre transform of a function $\rho: [0, \infty) \rightarrow \mathbb{R}_+$ is defined as $\rho^* = \sup\{xy - \rho(y) : y \geq 0\}$. Our answer to Question 2 is based on the following observation (which will also be used in the proof of Theorem 4).

Proposition 5. *For any Young functions φ, ψ satisfying Assumption (A), the function $\rho = \rho_{\varphi, \psi}$ is equivalent to η^* , where $\eta = (\psi^*)^{-1} \circ \varphi^*$. More precisely, for any $x \geq 0$,*

$$2\eta^*(2^{-1}x) \leq \rho(x) \leq 2^{-1}\eta^*(2x). \quad (6)$$

Before we prove the proposition let us derive the immediate corollary, whose optimality will also be shown in the next section.

Corollary 6. *Let ρ and ψ be two Young functions. Assume that ψ satisfies the assumption (A). Then for any Harris ergodic Markov chain, small set C , small measure ν and any $f: \mathcal{X} \rightarrow \mathbb{R}$ we have*

$$\|S\|_{\nu, \tilde{\varphi}} \leq 4m\|(\tau + 1)\|_{\nu, \psi}\|f\|_{\pi, \rho},$$

where $\tilde{\varphi} = (\psi^* \circ \rho^*)^*$.

Proof of Proposition 5. Using the fact that $\varphi^{**} = \varphi$ we get

$$\begin{aligned} \rho(x) &= \sup_{y \geq 0} \frac{\varphi(xy) - \psi(y)}{y} = \sup_{y \geq 0} \sup_{z \geq 0} \frac{xyz - \varphi^*(z) - \psi(y)}{y} \\ &= \sup_{z \geq 0} \left(xz - \inf_{y \geq 0} \frac{\varphi^*(z) + \psi(y)}{y} \right) = \tilde{\eta}^*(x), \end{aligned}$$

where $\tilde{\eta}(x) = \inf_{y \geq 0} (\varphi^*(z) + \psi(y))y^{-1}$. Note that as a function of y , $\varphi^*(z)y^{-1}$ decreases whereas $\psi(y)y^{-1}$ increases, so for all $z \geq 0$ we have

$$\frac{\varphi^*(z)}{y_0} \leq \eta(z) \leq 2\frac{\varphi^*(z)}{y_0},$$

where y_0 is defined by the equation $\varphi^*(z) = \psi(y_0)$, i.e. $y_0 = \psi^{-1}(\varphi(z))$. In combination with Lemma 21 from the Appendix, this yields

$$\frac{1}{2}\eta(z) \leq \tilde{\eta}(z) \leq 2\eta(z),$$

which easily implies that $2\eta^*(x/2) \leq \rho(x) \leq 2^{-1}\eta^*(2x)$ and thus ends the proof. \square

We also have the following Proposition whose prove is deferred to Section 3.

Proposition 7. *Let ψ and ρ be as in Corollary 6 and let φ be a Young function such that for every Markov chain (X_n) , small set C , small measure ν and $f: \mathcal{X} \rightarrow \mathbb{R}$ with $\|\tau\|_{\nu, \psi} < \infty$ and $\|f\|_{\pi, \rho} < \infty$ we have $\|S\|_{\nu, \varphi} < \infty$. Then $\varphi \preceq (\psi^* \circ \rho^*)^*$.*

Examples Let us now take a closer look at consequences of our theorems for classical Young functions. The following examples are straightforward and rely only on theorems presented in the last two sections and elementary formulas for Legendre transforms of classical Young functions. The formulas we present here will be used in Section 4. We also note that below we consider functions of the form $x \mapsto \exp(x^\alpha) - 1$ for $\alpha \in (0, 1)$. Formally such functions are not Young functions but it is easy to see that they can be modified for small values of x in such a way that they become Young functions. It is customary to define $\|X\|_{\psi_\alpha} = \inf\{C > 0: \mathbb{E} \exp((|X|/C)^\alpha) \leq 2\}$. Under such definition $\|\cdot\|_{\psi_\alpha}$ is a quasi-norm, which can be shown to be equivalent to the Orlicz norm corresponding to the modified function.

1. If $\varphi(x) = x^p$ and $\psi(x) = x^r$, where $r > p \geq 1$ then $\rho_{\varphi,\psi}(x) \simeq x^{\frac{p(r-1)}{r-p}}$.
2. If $\varphi(x) = \exp(x^\alpha) - 1$ and $\psi(x) = \exp(x^\beta) - 1$, where $\beta \geq \alpha$ then $\rho_{\varphi,\psi}(x) \simeq \exp(x^{\frac{\alpha\beta}{\beta-\alpha}}) - 1$.
3. If $\varphi(x) = x^p$ and $\psi(x) = \exp(x^\beta) - 1$, where $\beta > 0$ then $\rho_{\varphi,\psi}(x) \simeq x^p \log^{(p-1)/\beta} x$.
4. If $\psi(x) = x^r$ and $\rho(x) = x^p$ then $\varphi(x) \simeq x^{\frac{rp}{r+p-1}}$.
5. If $\psi(x) = \exp(x^\beta) - 1$ and $\rho(x) = \exp(x^\alpha) - 1$ ($\alpha, \beta > 0$), then $\varphi(x) \simeq \exp(x^{\frac{\alpha\beta}{\alpha+\beta}}) - 1$.
6. If $\psi(x) = \exp(x^\beta) - 1$ ($\beta > 0$) and $\rho(x) = x^p$ ($p \geq 1$), then $\varphi(x) \simeq \frac{x^p}{\log^{(p-1)/\beta} x}$.

2.2 The stationary case

We will now present answers to questions 1 and 2 in the stationary case. Let us start with the following

Definition 8. Let φ and ψ be Young functions. Assume that $\lim_{x \rightarrow 0} \psi(x)/x = 0$ and define the generalized Young function $\zeta = \zeta_{\varphi,\psi}$ by the formula

$$\zeta(x) = \sup_{y \geq 0} (\varphi(xy) - y^{-1}\psi(y)).$$

The function ζ will play in the stationary case a role analogous to the one of function ρ for the chain started from the small measure.

Theorem 9. Let φ and ψ be Young functions, $\lim_{x \rightarrow 0} \psi(x)/x = 0$. Let $\zeta = \zeta_{\varphi,\psi}$. Then for any Harris ergodic Markov chain (X_n) , small set C and small measure ν we have

$$\left\| \sum_{j=0}^{m\tau+m-1} f(X_j) \right\|_{\pi,\varphi} \leq m\|\tau + 1\|_{\nu,\psi} \left(1 + \delta\pi(C)\|\tau + 1\|_{\nu,\psi} \right) \|f\|_{\pi,\zeta}. \quad (7)$$

Proof. The proof is very similar to the proof of Theorem 2, however it involves one more use of Pitman's formula to pass from the stationary case to the case of the chain started from ν .

Consider any functional $F: \mathcal{X}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ (measurable wrt the product σ -field) on the space of the trajectories of the process $(X_n, Y_n)_{n \geq 0}$ (recall from the introduction that we identify X_n and \tilde{X}_n). By the definition of the split chain we have for any $i \in \mathbb{N}$,

$$\mathbb{E}(F((X_j)_{j \geq im}, (Y_j)_{j \geq i}) | \mathcal{F}_{im}^X, \mathcal{F}_i^Y) = G(X_{im}, Y_i),$$

where $G(x, y) = \mathbb{E}_{(x, y)} F((X_i)_{i \geq 0}, (Y_i)_{i \geq 0}) = \mathbb{E} F((X_i)_{i \geq 0}, (Y_i)_{i \geq 0} | X_0 = x, Y_0 = y)$. In particular for the functional

$$F((X_i)_{i \geq 0}, (Y_i)_{i \geq 0}) = \varphi((abm)^{-1} \sum_{i=0}^{m\tau+m-1} f(X_i)),$$

where $a = \|\tau + 1\|_{\nu, \psi}$ and $b = \|f\|_{\pi, \zeta}$, we have

$$\begin{aligned} \mathbb{E}_{\pi} \varphi((abm)^{-1} \sum_{i=0}^{m\tau+m-1} f(X_i)) &= \mathbb{E}_{\pi} G(X_0, Y_0) = \delta\pi(C) \mathbb{E}_{\nu} \sum_{i=0}^{\tau} G(X_{im}, Y_i) \\ &= \delta\pi(C) \sum_{i=0}^{\infty} \mathbb{E}_{\nu} G(X_{im}, Y_i) \mathbf{1}_{\{i \leq \tau\}} = \delta\pi(C) \sum_{i=0}^{\infty} \mathbb{E}_{\nu} \mathbb{E} \left(F((X_j)_{j \geq im}, (Y_j)_{j \geq i}) | \mathcal{F}_{im}^X, \mathcal{F}_i^Y \right) \mathbf{1}_{\{i \leq \tau\}} \\ &= \delta\pi(C) \sum_{i=0}^{\infty} \mathbb{E}_{\nu} \varphi \left((abm)^{-1} \sum_{j=im}^{m\tau+m-1} f(X_j) \right) \mathbf{1}_{\{i \leq \tau\}} = \delta\pi(C) \mathbb{E}_{\nu} \sum_{i=0}^{\tau} \varphi \left((abm)^{-1} \sum_{j=im}^{m\tau+m-1} f(X_j) \right) \\ &\leq \delta\pi(C) \mathbb{E}_{\nu} \sum_{i=0}^{\tau} \sum_{j=im}^{m\tau+m-1} \frac{1}{m(\tau-i+1)} \varphi((ab)^{-1}(\tau-i+1)f(X_j)) \\ &= \delta\pi(C) \mathbb{E}_{\nu} \sum_{j=0}^{m\tau+m-1} \sum_{i=0}^{\lfloor j/m \rfloor} \frac{1}{m(\tau-i+1)} \varphi((ab)^{-1}(\tau-i+1)f(X_j)) \\ &\leq \delta\pi(C) \mathbb{E}_{\nu} \sum_{j=0}^{m\tau+m-1} \frac{\lfloor jm^{-1} \rfloor + 1}{m(\tau+1)} \varphi((ab)^{-1}(\tau+1)f(X_j)), \end{aligned}$$

where the second equality follows from (2) and the two last inequalities from the convexity of φ .

We thus obtain

$$\begin{aligned} \mathbb{E}_{\pi} \varphi((abm)^{-1} S(f)) &\leq \delta\pi(C) m^{-1} \mathbb{E}_{\nu} \sum_{i=0}^{m\tau+m-1} \varphi((ab)^{-1}(\tau+1)f(X_i)) \\ &\leq \delta\pi(C) m^{-1} \mathbb{E}_{\nu} \sum_{i=0}^{m\tau+m-1} \zeta(b^{-1}f(X_i)) + \delta\pi(C) a \mathbb{E}_{\nu} \psi(a^{-1}(\tau+1)) \\ &\leq \mathbb{E}_{\pi} \zeta(b^{-1}f(X_0)) + \delta\pi(C) a \mathbb{E}_{\nu} \psi(a^{-1}(\tau+1)) \leq 1 + \delta\pi(C) a, \end{aligned}$$

which ends the proof. \square

Remark The dependence of the estimates presented in the above theorem on $\|\tau+1\|_{\nu,\psi}$ cannot be improved in the case of general Orlicz functions, since for $\varphi(x) = x$, $\psi(x) = x^2$, and $f \equiv 1$ we have $\|S(f)\|_{\pi,\varphi} = \mathbb{E}_\pi(\tau+1) \simeq \mathbb{E}_\nu(\tau+1)^2 = \|\tau+1\|_{\nu,\psi}^2$. However under additional assumptions on the growth of φ one can obtain a better estimate and replace the factor $1 + \delta\pi(C)\|\tau+1\|_{\nu,\psi}$ by $g(1 + \delta\pi(C)\|\tau+1\|_{\nu,\psi})$, where $g(r) = \sup_{x>0} x/\varphi^{-1}(\varphi(x)/r)$. For rapidly growing φ and large $\|\tau+1\|_{\nu,\psi}$ this may be an important improvement. It is also elementary to check that for $\phi(x) = \exp(x^\alpha) - 1$, we can use $g(r) \simeq \log^{1/\alpha}(r)$.

Just as in the case of Theorem 2, the estimates given in Theorem 9 are optimal. Below we state the corresponding optimality results, deferring their proofs to Section 3.

Proposition 10 (Weak optimality of Theorem 9). *Let φ and ψ be as in Theorem 9. Assume that a Young function ζ has the property that for every Harris ergodic chain (X_n) , small set C and small measure ν with $\|\tau\|_{\nu,\psi} < \infty$ and every function $f: \mathcal{X} \rightarrow \mathbb{R}$ such that $\|f\|_{\pi,\zeta} < \infty$, we have $\|S(f)\|_{\pi,\varphi} < \infty$. Then $\zeta_{\varphi,\psi} \preceq \zeta$.*

Theorem 11 (Strong optimality of Theorem 9). *Let φ, ψ and ζ be as in Theorem 9. Let $\tilde{\psi}(x) = \psi(x)/x$ and assume additionally that the function $\eta = \varphi^{-1} \circ \tilde{\psi}$ is equivalent to a Young function. Let Y be a random variable such that $\|Y\|_\zeta = \infty$. Then there exists a Harris ergodic Markov chain (X_n) on some Polish space \mathcal{X} with stationary distribution π , small set C , small measure ν and a function $f: \mathcal{X} \rightarrow \mathbb{R}$, such that the distribution of f under π is equal to the law of Y , $\|\tau\|_{\nu,\varphi} < \infty$ and $\|S(f)\|_{\pi,\varphi} = \infty$.*

Proposition 12. *For any Young functions φ, ψ such that $\lim_{x \rightarrow \infty} \psi(x)/x = 0$, the function $\zeta = \zeta_{\varphi,\psi}$ is equivalent to $\varphi \circ \eta^*$, where $\eta(x) = \varphi^{-1}(\psi(x)/x)$. More precisely, for all $x \geq 0$,*

$$\varphi(\eta^*(x)) \leq \zeta(x) \leq \frac{1}{2}\varphi(\eta^*(2x)).$$

Proof. Thanks to the assumption on ψ , we have $\lim_{y \rightarrow 0} \varphi(xy) - \psi(y)/y = 0$, so we can restrict our attention to $y > 0$, such that $\varphi(xy) > \psi(y)/y$ (note that if there are no such y , then $\eta^*(x) = \zeta(x) = 0$ and the inequalities of the proposition are trivially true). For such y , by convexity of φ we obtain

$$\varphi(xy) - \psi(y)/y \geq \varphi(xy - \varphi^{-1}(\psi(y)/y))$$

and

$$\varphi(xy) - \psi(y)/y \leq \varphi(xy) - \psi(y)/(2y) \leq \frac{1}{2}\varphi(2xy - \varphi^{-1}(\psi(y)/y)),$$

which, by taking the supremum over y , proves the proposition. \square

Lemma 13. *Assume that ζ and ψ are Young functions, $\tilde{\psi}(x) = \psi(x)/x$ is strictly increasing and $\tilde{\psi}(0) = 0$, $\tilde{\psi}(\infty) = \infty$. Let the function κ be defined by*

$$\kappa^{-1}(x) = \zeta^{-1}(x)\tilde{\psi}^{-1}(x) \tag{8}$$

for all $x \geq 0$. Then there exist constants $K, x_0 \in (0, \infty)$ such that for all $x \geq x_0$,

$$K^{-1}x \leq (\vartheta^*)^{-1}(x)\tilde{\psi}^{-1}(\kappa(x)) \leq 2x \quad (9)$$

where $\vartheta = \kappa^{-1} \circ \tilde{\psi}$.

Moreover the function $\tilde{\zeta} = \kappa \circ \vartheta^*$ is equivalent to ζ .

Proof. Note first that $\vartheta(x) = \zeta^{-1}(\tilde{\psi}(x))x$, and so ϑ is equivalent to a Young function (e.g. by Lemma 20 in the Appendix). The inequalities (9) follow now by Lemma 21 from the Appendix.

Moreover

$$\tilde{\zeta}^{-1}(x) = (\vartheta^*)^{-1}(\kappa^{-1}(x))$$

and thus by (9) for x sufficiently large,

$$K^{-1}\zeta^{-1}(x) = K^{-1}\frac{\kappa^{-1}(x)}{\tilde{\psi}^{-1}(x)} \leq \tilde{\zeta}^{-1}(x) \leq 2\frac{\kappa^{-1}(x)}{\tilde{\psi}^{-1}(x)} = 2\zeta^{-1}(x),$$

which clearly implies that $\tilde{\zeta}(K^{-1}x) \leq \zeta(x) \leq \tilde{\zeta}(2x)$ for x large enough. \square

If now φ is a Young function such that $\varphi \preceq \kappa$, then $\varphi((\varphi^{-1} \circ \tilde{\psi})^*) \preceq \kappa((\kappa^{-1} \circ \tilde{\psi})^*) \simeq \zeta$. Thus the above Lemma, together with Theorem 9 and Proposition 12 immediately gives the following

Corollary 14. *Assume that ζ and ψ are Young functions, $\tilde{\psi}(x) = \psi(x)/x$ is strictly increasing, $\tilde{\psi}(0) = 0$, $\tilde{\psi}(\infty) = \infty$. Let the function κ be defined by (8). If φ is a Young function such that $\varphi \preceq \kappa$, then there exists $K < \infty$, such that for any Harris ergodic Markov chain (X_n) on \mathcal{X} , small set C , small measure ν and $f: \mathcal{X} \rightarrow \mathbb{R}$,*

$$\|S(f)\|_{\pi, \varphi} \leq K\|\tau + 1\|_{\nu, \psi} \left(1 + \delta\pi(C)\|\tau + 1\|_{\nu, \psi}\right) \|f\|_{\pi, \zeta}. \quad (10)$$

Remark For slowly growing functions ψ and ζ there may be no Orlicz function φ such that $\varphi \preceq \kappa$. This is not surprising since as we will see from the construction presented in Section 3.1 the π -integrability of $S(f)$ is closely related to integrability of functions from a point-wise product of Orlicz spaces. In consequence $S(f)$ may not even be integrable.

We have the following optimality result corresponding to Corollary 14. Its proof will be presented in the next section.

Proposition 15. *Assume that ζ and ψ are Young functions, $\tilde{\psi}(x) = \psi(x)/x$ is strictly increasing, $\tilde{\psi}(0) = 0$, $\tilde{\psi}(\infty) = \infty$. Let the function κ be defined by (8) and let φ be a Young function such that for every ergodic Markov chain (X_n) , small set C , small measure ν and $f: \mathcal{X} \rightarrow \mathbb{R}$ with $\|\tau\|_{\nu, \psi} < \infty$ and $\|f\|_{\pi, \zeta} < \infty$, we have $\|S(f)\|_{\pi, \varphi} < \infty$. Then $\varphi \preceq \kappa$.*

Remark By convexity of φ , the condition $\varphi \preceq \kappa$ holds iff there exists a constant $K < \infty$ and $x_0 > 0$, such that

$$K\varphi^{-1}(x) \geq \tilde{\psi}^{-1}(x)\zeta^{-1}(x)$$

for $x > x_0$. Thus under the assumptions that $\tilde{\psi}$ is strictly increasing $\tilde{\psi}(0) = 0$, $\tilde{\psi}(\infty) = \infty$, the above condition characterizes the triples of Young functions such that $\|f\|_{\pi, \zeta} < \infty$ implies $\|S(f)\|_{\pi, \varphi} < \infty$ for all Markov chains with $\|\tau\|_{\nu, \psi} < \infty$.

Examples Just as in the previous section we will now present some concrete formulas for classical Young functions, some of which will be used in Section 4 to derive tail inequalities for additive functionals of stationary Markov chains.

1. If $\varphi(x) = x^p$ and $\psi(x) = x^r$, where $r > p + 1 \geq 2$ then $\zeta_{\varphi,\psi}(x) \simeq x^{\frac{p(r-1)}{r-p-1}}$.
2. If $\varphi(x) = \exp(x^\alpha) - 1$ and $\psi(x) = \exp(x^\beta) - 1$, where $\beta \geq \alpha$ then $\zeta_{\varphi,\psi}(x) \simeq \exp(x^{\frac{\alpha\beta}{\beta-\alpha}}) - 1$.
3. If $\varphi(x) = x^p$ and $\psi(x) = \exp(x^\beta) - 1$, where $\beta > 0$ then $\zeta_{\varphi,\psi}(x) \simeq x^p \log^{p/\beta} x$.
4. If $\psi(x) = x^r$ and $\zeta(x) = x^p$ ($r \geq 2, p \geq (r-1)/(r-2)$) then $\varphi(x) \simeq x^{\frac{(r-1)p}{r+p-1}}$.
5. If $\psi(x) = \exp(x^\beta) - 1$ and $\zeta(x) = \exp(x^\alpha) - 1$ ($\alpha, \beta > 0$), then $\varphi(x) \simeq \exp(x^{\frac{\alpha\beta}{\alpha+\beta}}) - 1$.
6. If $\psi(x) = \exp(x^\beta) - 1$ ($\beta > 0$) and $\zeta(x) = x^p$ ($p > 1$), then $\varphi(x) \simeq \frac{x^p}{\log^{p/\beta} x}$.

3 Proofs of optimality

3.1 Main counterexample

We will now introduce a general construction of a Markov chain which will serve as an example in proofs of all our optimality theorems.

Let \mathcal{S} be a Polish space and let α be a Borel probability measure on \mathcal{S} . Consider two functions $\tilde{f}: \mathcal{S} \rightarrow \mathbb{R}$ and $h: \mathcal{S} \rightarrow \mathbb{N} \setminus \{0\}$. We will construct a Markov chain on some Polish space $\mathcal{X} \supseteq \mathcal{S}$, a small set $C \subseteq \mathcal{X}$, a probability measure ν and a function $f: \mathcal{X} \rightarrow \mathbb{R}$, possessing the following properties.

Properties of the chain

- (i) The condition (1) is satisfied with $m = 1$ and $\delta = 1$ (in other words C is an atom for the chain),
- (ii) $\nu(\mathcal{S}) = 1$,
- (iii) for any $x \in \mathcal{S}$, $\mathbb{P}_x(\tau + 1 = h(x)) = 1$,
- (iv) for any $x \in \mathcal{S}$, $\mathbb{P}_x(S(f) = \tilde{f}(x)h(x)) = 1$,
- (v) For any function $G: \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\mathbb{E}_\nu G(S(f)) = R \int_{\mathcal{S}} G(\tilde{f}(x)h(x))h(x)^{-1}\alpha(dx)$$

and

$$\mathbb{E}_\nu G(\tau + 1) = R \int_{\mathcal{S}} G(h(x))h(x)^{-1}\alpha(dx),$$

where $R = (\int_C h(y)^{-1}\alpha(dy))^{-1}$,

(vi) (X_n) admits a unique stationary distribution π and the law of f under π is the same as the law of \tilde{f} under α ,

(vii) for any nondecreasing function $F: \mathcal{X} \rightarrow \mathbb{R}$,

$$\mathbb{E}_\pi F(|S(f)|) \geq \frac{1}{2} \int_{\mathcal{S}} F(h(x)|\tilde{f}(x)|/2) \alpha(dx).$$

(viii) if $\alpha(\{x: h(x) = 1\}) > 0$ then the chain is Harris ergodic.

Construction of the chain Let $\mathcal{X} = \bigcup_{n=1}^{\infty} \{x \in \mathcal{S}: h(x) \geq n\} \times \{n\}$. As a disjoint union it clearly possesses a natural structure of a Polish space inherited from \mathcal{S} . Formally $\mathcal{S} \not\subseteq \mathcal{X}$ but it does not pose a problem as we can clearly identify \mathcal{S} with $\mathcal{S} \times \{1\} = \{x \in \mathcal{S}: h(x) \geq 1\} \times \{1\}$.

The dynamics of the chain will be very simple.

- If $X_n = (x, i)$ and $h(x) > i$, then with probability one $X_{n+1} = (x, i + 1)$.
- If $X_n = (x, i)$ and $h(x) = i$, then $X_{n+1} = (y, 1)$, where y is distributed according to the probability measure

$$\nu(dy) = Rh(y)^{-1} \alpha(dy). \quad (11)$$

More formally, the transition function of the chain is given by

$$P((x, i), A) = \begin{cases} \delta_{(x, i+1)}(A) & \text{if } i < h(x) \\ \nu(\{y \in \mathcal{S}: (y, 1) \in A\}) & \text{if } i = h(x). \end{cases}$$

In other words the chain describes a particle, which after departing from a point $(x, 1) \in \mathcal{S}$ changes its 'level' by jumping deterministically to points $(x, 2), \dots, (x, h(x))$ and then goes back to 'level' one by selecting the first coordinate according to the measure ν .

Clearly $\nu(\mathcal{S}) = 1$ and so condition (ii) is satisfied. Note that α and ν are formally measures on \mathcal{S} , but we may and will sometimes treat them as measures on \mathcal{X} .

Let now $C = \{(x, i) \in \mathcal{X}: h(x) = i\}$. Then $P((x, i), A) = \nu(A)$ for any $(x, i) \in C$ and a Borel subset A of \mathcal{X} , which shows that (1) holds with $m = 1$ and $\delta = 1$.

Let us now prove condition (iii). Since C is an atom for the chain, $Y_n = 1$ iff $X_n \in C$. Moreover if $X_0 = (x, 1) \simeq x \in \mathcal{S}$, then $X_i = (x, i + 1)$ for $i + 1 \leq h(x)$ and $\tau = \inf\{i \geq 0: X_i \in C\} = \inf\{i \geq 0: i + 1 = h(x)\} = h(x) - 1$, which proves property (iii).

To assure that property (iv) holds it is enough to define

$$f((x, i)) = \tilde{f}(x),$$

since then $f(X_0) = (x, 1)$ implies that $f(X_n) = \tilde{f}(x)$ for $n \leq \tau$.

Condition (v) follows now from properties (ii), (iii) and (iv) together with formula (11).

We will now pass to conditions (vi) and (vii).

By the construction of the chain it is easy to prove that the chain admits a unique stationary measure π given by

$$\pi(A \times \{k\}) = \alpha(A)n^{-1}$$

for $A \subseteq \{x \in \mathcal{S} : h(x) = n\}$ and any $k \leq n$. Thus for any Borel set $B \subseteq \mathbb{R}$ we have

$$\begin{aligned} \pi(\{(x, i) \in \mathcal{X} : f((x, i)) \in B\}) &= \pi(\{(x, i) \in \mathcal{X} : \tilde{f}(x) \in B\}) \\ &= \sum_{n \geq 1} \pi(\{(x, i) \in \mathcal{X} : h(x) = n, \tilde{f}(x) \in B\}) \\ &= \sum_{n \geq 1} n \cdot n^{-1} \alpha(\{x \in \mathcal{S} : h(x) = n, \tilde{f}(x) \in B\}) \\ &= \alpha(\{x \in \mathcal{S} : \tilde{f}(x) \in B\}). \end{aligned}$$

As for (vii), $X_0 = (x, i)$ implies that

$$S(f) = (h(x) - i + 1)\tilde{f}(x).$$

Thus, letting $A_{n,k} = \{(x, k) \in \mathcal{X} : h(x) = n\}$, $B_n = \{x \in \mathcal{S} : h(x) = n\}$, we get

$$\begin{aligned} \mathbb{E}_\pi F(|S(f)|) &= \int_{\mathcal{X}} F((h(x) - i + 1)|\tilde{f}(x)|) \pi(d(x, i)) = \sum_{n,k} \int_{A_{n,k}} F((n - k + 1)|\tilde{f}(x)|) \pi(d(x, i)) \\ &= \sum_n \sum_{k \leq n} \int_{B_n} n^{-1} F((n - k + 1)|\tilde{f}(x)|) \alpha(dx) \geq \sum_n \int_{B_n} \frac{1}{2} F(n|\tilde{f}(x)|/2) \alpha(dx) \\ &= \frac{1}{2} \int_{\mathcal{S}} F(h(x)|\tilde{f}(x)|/2) \alpha(dx), \end{aligned}$$

proving (vii).

Now we will prove (viii). Note that $A := \{x \in \mathcal{S} : h(x) = 1\} \subseteq C$. Thus if $\alpha(A) > 0$ then also $\nu(C) > 0$, which proves that the chain is strongly aperiodic (see e.g. chapter 5 of [26] or Chapter 2 of [29]). Moreover one can easily see that π is an irreducibility measure for the chain and the chain is Harris recurrent. Thus by Proposition 6.3. of [29] the chain is Harris ergodic (in fact in [29] ergodicity is defined as aperiodicity together with positiveness and Harris recurrence and Proposition 6.3. states that this is equivalent to convergence of n -step probabilities for any initial point).

What remains to be proven is condition (i). Since $\pi(C) > 0$ we have $C \in \mathcal{E}^+$, whereas inequality (1) for $m = \delta = 1$ is satisfied by the construction.

3.2 The chain started from ν

We will start with the proof of Proposition 3. The chain constructed above will allow us to reduce it to elementary techniques from the theory of Orlicz spaces.

Proof of Proposition 3. Assume that the function ρ does not satisfy the condition $\rho_{\varphi, \psi} \preceq \rho$. Thus there exists a sequence of numbers $x_n \rightarrow \infty$ such that

$$\rho(x_n) < \rho_{\varphi, \psi}(x_n 2^{-n}).$$

By the definition of $\rho_{\varphi,\psi}$ this means that there exists a sequence $t_n > 0$ such that

$$\frac{\varphi(x_n t_n 2^{-n})}{t_n} \geq \rho(x_n) + \frac{\psi(t_n)}{t_n}.$$

One can assume that $t_n \geq 2$. Indeed, for all n large enough if $t_n \leq 2$, then

$$\frac{\varphi((x_n 2^{-1}) 2^{-(n-1)} \cdot 2)}{2} \geq \frac{\varphi(x_n t_n 2^{-n})}{t_n} \geq \rho(x_n) \geq 2\rho(x_n 2^{-1}) \geq \rho(x_n 2^{-1}) + \frac{\psi(2)}{2}.$$

Set $\tau_n = \lfloor t_n \rfloor$ for $n \geq 1$ and $\tau_0 = 1$. We have for $n \geq 1$

$$\frac{\varphi(x_n \tau_n 2^{1-n})}{\tau_n} \geq \frac{\varphi(x_n t_n 2^{-n})}{t_n} \geq \rho(x_n) + \frac{\psi(t_n)}{t_n} \geq \rho(x_n) + \frac{\psi(\tau_n)}{\tau_n} \geq 1, \quad (12)$$

where in the last inequality we used assumption (A). Define now $p_n = C 2^{-n} (\psi(\tau_n)/\tau_n + \rho(x_n))^{-1}$, where C is a constant such that $\sum_{n \geq 0} p_n = 1$. Consider a Polish space \mathcal{S} with a probability measure α , a partition $\mathcal{S} = \bigcup_{n \geq 0} A_n$, $\alpha(A_n) = p_n$ and two functions h and \tilde{f} , such that $\tilde{f}(x) = x_n$ and $h(x) = \tau_n$ for $x \in A_n$.

Let $(X_n)_{n \geq 0}$ be the Markov chain obtained by applying to \mathcal{S} , \tilde{f} and h the main construction introduced in Section 3.1. By property (viii) and the condition $\tau_0 = 1$, the chain is Harris ergodic. By property (v) we have

$$\mathbb{E}_\nu \psi(\tau + 1) = R \int_{\mathcal{S}} \psi(h(x)) h(x)^{-1} \alpha(dx) = R \sum_{n \geq 0} \frac{\psi(\tau_n)}{\tau_n} p_n \leq 2RC$$

by the definition of p_n . Thus the chain (X_n) satisfies $\|\tau\|_{\nu,\psi} < \infty$.

By property (vi) we get

$$\mathbb{E}_\pi \rho(f) = \int_{\mathcal{S}} \rho(\tilde{f}(x)) \alpha(dx) = \sum_{n \geq 0} \rho(x_n) p_n \leq 2C.$$

On the other hand for any $\theta > 0$ we have by property (v), the construction of functions \tilde{f}, g and (12),

$$\begin{aligned} \mathbb{E}_\nu \varphi(\theta |S(f)|) &= R \int_{\mathcal{S}} \varphi(\theta |\tilde{f}(x)| h(x)) h(x)^{-1} \alpha(dx) \\ &\geq R \sum_{n \geq 1} \frac{\varphi(2^{n-1} \theta x_n \tau_n 2^{1-n})}{\tau_n} p_n \geq R \sum_{2^{n-1} \theta \geq 1} 2^{n-1} \theta \frac{\varphi(x_n \tau_n 2^{1-n})}{\tau_n} p_n \\ &\geq R \sum_{2^{n-1} \theta \geq 1} 2^{n-1} \theta \left(\rho(x_n) + \frac{\psi(\tau_n)}{\tau_n} \right) p_n = \infty, \end{aligned}$$

which shows that $\|S(f)\|_{\nu,\varphi} = \infty$ and proves the proposition. \square

Proof of Theorem 4. Let \mathcal{S} be a Polish space, α a probability measure on \mathcal{S} and $\tilde{f}: \mathcal{S} \rightarrow \mathbb{R}$ a function whose law under α is the same as the law of Y .

We will consider in detail only the case when $\lim_{x \rightarrow \infty} \varphi(x)/x = \infty$. It is easy to see using formula (3) and the construction below that in the case $\varphi \simeq id$ the theorem also holds (note that in this case also $\rho \simeq id$).

By the convexity assumption and Lemma 22 from the Appendix, we obtain that $\eta = (\psi^*)^{-1} \circ \varphi^*$ is equivalent to a Young function. Thus by Proposition 5 and Lemma 21 from the Appendix we get

$$\rho^*(\cdot) \simeq (\psi^*)^{-1} \circ \varphi^*(\cdot) \simeq \frac{\varphi^*(\cdot)}{\psi^{-1} \circ \varphi^*(\cdot)}. \quad (13)$$

By Lemma 19 in the Appendix (or in the case when $\rho^* \simeq id$ by the well known facts about the spaces L_1 and L_∞), there exists a function $g: \mathcal{S} \rightarrow \mathbb{R}_+$ such that

$$\int_{\mathcal{S}} \frac{\varphi^*(g(x))}{\psi^{-1}(\varphi^*(g(x)))} \alpha(dx) < \infty \text{ and } \int_{\mathcal{S}} |\tilde{f}(x)| g(x) \alpha(dx) = \infty. \quad (14)$$

Define the function $h: \mathcal{S} \rightarrow \mathbb{N} \setminus \{0\}$ by $h(x) = \lfloor \psi^{-1}(\varphi^*(g(x))) \rfloor + 1$.

Let now \mathcal{X} , (X_n) and f be the Polish space, Markov chain and function obtained from $\mathcal{S}, \alpha, \tilde{f}, h$ according to the main construction of Section 3.1. Note that we can assume that $\alpha(\{x: h(x) = 1\}) > 0$ and thus by property (viii) this chain is Harris ergodic.

Note that by the definition of h , if $h(x) \geq 2$ then $h(x) \leq 2\psi^{-1}(\varphi^*(g(x)))$. Thus by property (v) and (14) we get

$$\begin{aligned} \mathbb{E}_\nu \psi((\tau + 1)/2) &= R \int_{\mathcal{S}} \psi(h(x)/2) h(x)^{-1} \alpha(dx) \\ &\leq R\psi(1/2) + R \int_{\mathcal{S}} \frac{\varphi^*(g(x))}{\psi^{-1}(\varphi^*(g(x)))} \alpha(dx) < \infty, \end{aligned}$$

which implies that $\|\tau\|_{\nu, \psi} < \infty$. Recall now the definition of ν given in (11). For all $a > 0$ we have

$$\mathbb{E}_\nu \varphi(S(f)/a) = \int_{\mathcal{S}} \varphi(\tilde{f}(x)h(x)/a) \nu(dx),$$

which implies that $\|S(f)\|_{\nu, \varphi} < \infty$ iff $\|\tilde{f}h\|_{\nu, \varphi} < \infty$ (note that on the left hand side ν is treated as a measure on \mathcal{X} and on the right hand side as a measure on \mathcal{S}).

Note however that by (14) we have

$$\int_{\mathcal{S}} \varphi^*(g(x)) \nu(dx) = R \int_{\mathcal{S}} \frac{\varphi^*(g(x))}{h(x)} \alpha(dx) \leq R \int_{\mathcal{S}} \frac{\varphi^*(g(x))}{\psi^{-1}(\varphi^*(g(x)))} \alpha(dx) < \infty,$$

which gives $\|g\|_{\nu, \varphi^*} < \infty$, but

$$\int_{\mathcal{S}} \tilde{f}(x)h(x)g(x) \nu(dx) = R \int_{\mathcal{S}} \tilde{f}(x)g(x) \alpha(dx) = \infty.$$

This shows that $\|\tilde{f}h\|_{\nu, \varphi} = \infty$ and ends the proof. \square

Proof of Proposition 7. Note that for any function f (not necessarily equivalent to a Young function) we have $f^{**} \leq f$. Thus by Proposition 5 we have

$$\rho_{\varphi, \psi} \preceq \rho \iff ((\psi^*)^{-1} \circ \varphi^*)^* \preceq \rho \implies \rho^* \preceq (\psi^*)^{-1} \circ \varphi^* \iff \psi^* \circ \rho^* \preceq \varphi^* \iff \varphi \preceq (\psi^* \circ \rho^*)^*,$$

which ends the proof by Proposition 3. \square

3.3 The stationary case

For the proofs of results concerning optimality of our estimates for the chain started from π , we will also use the general construction from Section 3.1. As already mentioned, in this case the problem turns out to be closely related to the classical theory of point-wise multiplication of Orlicz spaces (we refer to [32] for an overview)

Proof of Proposition 10. Assume that the function ζ does not satisfy the condition $\zeta_{\varphi, \psi} \preceq \zeta$. Thus there exists a sequence of numbers $x_n \rightarrow \infty$, such that $\zeta(x_n) < \zeta_{\varphi, \psi}(x_n 2^{-n})$, i.e. for some sequence $t_n > 0$, $n = 1, 2, \dots$, we have

$$\varphi(x_n 2^{-n} t_n) \geq \zeta(x_n) + \psi(t_n)/t_n.$$

Similarly as in the proof of Proposition 3, we show that without loss of generality we can assume that t_n are positive integers and thus the right hand side above is bounded from below. Let us additionally define $t_0 = 1$.

Let $p_n = C 2^{-n} (\zeta(x_n) + \psi(t_n)/t_n)^{-1}$, where C is such that $\sum_{n \geq 0} p_n = 1$ and consider a probability space (\mathcal{S}, σ) , where $\mathcal{S} = \bigcup_n A_n$ with A_n disjoint and $\alpha(A_n) = p_n$ together with two functions $\tilde{f}: \mathcal{S} \rightarrow \mathbb{R}$ and $h: \mathcal{S} \rightarrow \mathbb{R}$ such that for $x \in A_n$, we have $\tilde{f}(x) = x_n$, $h(x) = t_n$.

By applying to \mathcal{S}, \tilde{f} and h the general construction of Section 3.1, we get a Harris ergodic Markov chain and a function f , which by properties (v) and (vi) satisfy

$$\begin{aligned} \mathbb{E}_\nu \psi(\tau + 1) &= R \sum_{n \geq 0} \frac{\psi(t_n)}{t_n} p_n \leq 2RC, \\ \mathbb{E}_\pi \zeta(f) &= \sum_{n \geq 0} \zeta(x_n) p_n \leq C. \end{aligned}$$

But by property (vii) we get for any $\theta > 0$,

$$\begin{aligned} \mathbb{E}_\pi \varphi(\theta |S(f)|) &\geq \frac{1}{2} \int_{\mathcal{S}} \varphi(\theta h(x) |\tilde{f}(x)|/2) \alpha(dx) = \frac{C}{2} \sum_{n \geq 0} \varphi(\theta x_n t_n/2) p_n \\ &\geq \frac{C}{2} \sum_{n \geq 1} \varphi(2^{n-1} \theta x_n t_n 2^{-n}) p_n \geq \frac{C}{2} \sum_{2^{n-1} \theta \geq 1} 2^{n-1} \theta \varphi(x_n t_n 2^{-n}) p_n \\ &\geq \frac{C}{2} \sum_{2^{n-1} \theta \geq 1} 2^{n-1} \theta (\zeta(x_n) + \psi(t_n)/t_n) p_n = \infty, \end{aligned}$$

which ends the proof. \square

Proof of Theorem 11. Consider first the case $\lim_{x \rightarrow \infty} \eta(x)/x = \infty$.

We will show that for some constant C and x large enough we have

$$\varphi^{-1}(x) \leq C\zeta^{-1}(x)\tilde{\psi}^{-1}(x). \quad (15)$$

We have $\eta^{-1}(x) = \tilde{\psi}^{-1}(\varphi(x))$ and thus by the assumption on η and Lemma 21 from the Appendix, we get $\varphi^{-1}(x) \leq C(\eta^*)^{-1}(\varphi^{-1}(x))\tilde{\psi}^{-1}(x)$ for some constant $C < \infty$ and x large enough. But by Proposition 12, $(\eta^*)^{-1}(\varphi^{-1}(x)) \leq 2\zeta^{-1}(x)$ and thus (15) follows.

If $\lim_{x \rightarrow \infty} \eta(x)/x < \infty$ then (15) also holds if we interpret ζ^{-1} as the generalized inverse (note that in this case $L_\zeta = L_\infty$)

Theorem 1 from [25] states that if $\varphi, \zeta, \tilde{\psi}$ are Young functions such that (15) holds for all $x \in [0, \infty)$ and Y is a random variable such that $\|Y\|_\zeta = \infty$, then there exists a random variable X , such that $\|X\|_{\tilde{\psi}} < \infty$ and $\|XY\|_\varphi = \infty$. One can easily see that the functions $\varphi, \zeta, \tilde{\psi}$ can be modified (for small values of x) to Young functions such that (15) holds for all $x \geq 0$. Thus there exists X satisfying the above condition. Clearly one can assume that with probability one X is a positive integer and $\mathbb{P}(X = 1) > 0$. Consider now a Polish space (\mathcal{S}, α) and $\tilde{f}, h: \mathcal{S} \rightarrow \mathbb{R}$ such that (\tilde{f}, h) is distributed like (Y, X) . Let (X_n) be the Markov chain given by the construction of Section 3.1. By property (v) we have

$$\mathbb{E}_\nu \psi\left(\frac{\tau+1}{a}\right) = R \int_{\mathcal{S}} \psi\left(\frac{h(x)}{a}\right) h(x)^{-1} \alpha(dx) = \frac{R}{a} \mathbb{E} \tilde{\psi}\left(\frac{X}{a}\right) < \infty$$

for a large enough, since $\|X\|_{\tilde{\psi}} < \infty$. By property (vi), the law of f under π is equal to the law of Y . Finally, by property (vii), for every $a > 0$,

$$\mathbb{E}_\pi \varphi\left(\frac{|S(f)|}{a}\right) \geq 2^{-1} \mathbb{E} \varphi\left(\frac{XY}{2a}\right) = \infty,$$

which proves that $\|S(f)\|_{\pi, \varphi} = \infty$. □

Proof of Proposition 15. Let $\eta = \varphi^{-1} \circ \tilde{\psi}$. By Propositions 10, 12 and Lemma 13 we have

$$\varphi \circ \eta^* \preceq \zeta \simeq \kappa \circ \vartheta^*,$$

where $\vartheta = \kappa^{-1} \circ \tilde{\psi}$. Thus $(\vartheta^*)^{-1} \circ \kappa^{-1} \preceq (\eta^*)^{-1} \circ \varphi^{-1}$. Another application of Lemma 13 together with Lemma 21 in the Appendix yield for some constant $C \in (1, \infty)$ and x large enough,

$$\begin{aligned} \kappa^{-1}(x) &\leq C(\vartheta^*)^{-1}(\kappa^{-1}(x))\tilde{\psi}^{-1}(x) \leq C^2(\eta^*)^{-1}(\varphi^{-1}(Cx))\tilde{\psi}^{-1}(Cx) \\ &= C^2(\eta^*)^{-1}(\varphi^{-1}(Cx))\eta^{-1}(\varphi^{-1}(Cx)) \leq 2C^2\varphi^{-1}(Cx) \end{aligned}$$

which implies that $\varphi \preceq \kappa$. □

4 Applications

4.1 Limit theorems for additive functionals

It is well known that for a Harris ergodic Markov chain and a function f , the CLT

$$\frac{f(X_0) + \dots + f(X_{n-1})}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma_f^2) \quad (16)$$

holds in the stationary case iff it holds for any initial distribution.

Moreover (see [12] and [6]) under the assumption that $\mathbb{E}_\pi f^2 < \infty$, the above CLT holds iff $\mathbb{E}_\nu S(f) = 0$, $\mathbb{E}_\nu (S(f))^2 < \infty$ and the asymptotic variance is given by $\sigma_f^2 = \delta\pi(C)m^{-1}(\mathbb{E}_{S_1}(f)^2 + 2\mathbb{E}_{S_1}(f)s_2(f))$. If the chain has an atom, this equivalence holds without the assumption $\mathbb{E}_\pi f^2 < \infty$.

It is also known (see [12]) that the condition $\mathbb{E}_\pi f = 0$, $\mathbb{E}_\nu S(|f|)^2 < \infty$ implies the law of the iterated logarithm

$$-\sigma_f = \liminf_{n \rightarrow \infty} \frac{\sum_{i=0}^{n-1} f(X_i)}{\sqrt{n \log \log n}} \leq \limsup_{n \rightarrow \infty} \frac{\sum_{i=0}^{n-1} f(X_i)}{\sqrt{n \log \log n}} = \sigma_f \text{ a.s.} \quad (17)$$

Moreover for chains with an atom $\limsup_{n \rightarrow \infty} \frac{|\sum_{i=0}^{n-1} f(X_i)|}{\sqrt{n \log \log n}} < \infty$ a.s. implies the CLT (see [12], Theorem 2.2. and Remark 2.3).

Our results from section 2.1 can be thus applied to give optimal conditions for CLT and LIL in terms of ergodicity of the chain (expressed by Orlicz integrability of the regeneration time) and integrability of f wrt the stationary measure.

The following Theorem is an immediate consequence of Theorems 2, 4 and Proposition 3.

Theorem 16. *Consider a Harris ergodic Markov chain (X_n) on a Polish space \mathcal{X} and a function $f: \mathcal{X} \rightarrow \mathbb{R}$, $\mathbb{E}_\pi f = 0$. Let ψ be a Young function such that $\lim_{x \rightarrow 0} \psi(x)/x = 0$ and assume that $\|\tau\|_{\nu, \psi} < \infty$. Let finally $\rho(x) = \tilde{\psi}^*(x^2)$, where $\tilde{\psi}(x) = \psi(x)/x$. If $\|f\|_{\pi, \rho} < \infty$ then the CLT (16) and LIL (17) hold.*

Moreover every Young function $\tilde{\rho}$ such that $\|f\|_{\pi, \tilde{\rho}}$ implies CLT (or LIL) for all Harris ergodic Markov chains with $\|\tau\|_{\nu, \psi} < \infty$ satisfies $\rho \preceq \tilde{\rho}$.

If the function $x \mapsto \sqrt{\psi(x)}$ is equivalent to a Young function then for every random variable Y with $\|Y\|_\rho = \infty$ one can construct a stationary Harris ergodic Markov chain (X_n) and a function f such that $f(X_n)$ has the same law as Y , $\|\tau\|_{\nu, \psi} < \infty$ and both (16) and (17) fail.

Remark As noted in [20] in the case of geometric ergodicity, i.e. when $\psi(x) = \exp(x) - 1$, the CLT part of the above theorem can be obtained from results in [16], giving very general and optimal conditions for CLT under α -mixing. The integrability condition for f is in this case $\mathbb{E}_\pi f^2 \log_+(|f|) < \infty$. The sufficiency for the LIL part can be similarly deduced from [36].

The equivalence of the exponential decay of mixing coefficients with ergodicity of Markov chains (measured in terms of ψ) follows from [30, 31]. Optimality of the condition follows from examples given in [10]. Examples of geometrically ergodic Markov chains and a function f such that $\mathbb{E} f^2 < \infty$ and the CLT fails have been also constructed in [18, 11]. Let us point out that if the Markov chain is reversible and geometrically ergodic, then $\|f\|_{\pi, 2} < \infty$ implies the CLT and thus also $\mathbb{E}_\nu S(f)^2 < \infty$. Thus under this additional assumptions our formulas for $\psi(x) = \exp(x) - 1$ and $\rho(x) = x^2$ are no longer optimal (our example from Section 3.1 is obviously non-reversible). It would be of interest to derive counterparts of theorems from Section 2 under the assumption of reversibility.

It is possible that in a more general case Theorem 16 can also be recovered from the above results, by proper characterizations of ergodicity in terms of mixing and characterizations of Orlicz spaces in terms of some weighted inequalities involving the tail of the function. However

we have not attempted to do this in full generality (we have only verified that such an approach works in the case of $\psi(x) = x^p$).

Let us also remark that to our best knowledge, so far there has been no 'regeneration' proof of Theorem 16 even in the case of geometric ergodicity.

Berry-Esseen type theorems Similarly we can use a result by Bolthausen [8, 9] to derive Berry-Esseen type bounds for additive functionals of stationary chains. More specifically Lemma 2 in [9], together with Theorem 2 give

Theorem 17. *Let (X_n) be a stationary strongly aperiodic Harris ergodic Markov chain on \mathcal{X} , such that $\|\tau\|_{\nu, \psi} < \infty$, where ψ is a Young function satisfying $(x \mapsto x^3) \preceq \psi$ and $\lim_{x \rightarrow 0} \psi(x)/x = 0$. Let $\rho = \Psi^*(x^3)$, where $\Psi(x) = \psi(\sqrt{x})/\sqrt{x}$. Then for every $f: \mathcal{X} \rightarrow \mathbb{R}$ such that $\|f\|_{\pi, \rho} < \infty$ and $\sigma_f^2 := \mathbb{E}(S(f))^2 > 0$ we have*

$$\left| \mathbb{P}\left(\frac{\sum_{i=0}^{n-1} f(X_i) - \mathbb{E}_\pi f}{\sigma_f \sqrt{n}}\right) - \Phi(x) \right| = \mathcal{O}(n^{-1/2}),$$

where $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x \exp(-y^2/2) dy$.

4.2 Tail estimates

The last application we develop concerns tail inequalities for additive functionals. The approach we take is by now fairly standard (see e.g. [13, 15, 7, 1, 2, 22, 23]) and relies on splitting the additive functional into a sum of independent (or one-dependent blocks) and using inequalities for sums of independent random variables. Our results on Orlicz integrability imply inequalities for the chain started from the small measure (an atom) or from the stationary distribution. The former case may have potential applications in MCMC algorithms in situations when small measure is known explicitly and one is able to sample from it.

In what follows we denote $\psi_\alpha = \exp(x^\alpha) - 1$.

Theorem 18. *Let $(X_n)_{n \geq 0}$ be a Harris ergodic Markov chain on \mathcal{X} . Assume that $\|\tau\|_{\nu, \psi_\alpha} < \infty$ for some $\alpha \in (0, 1)$. Let $f: \mathcal{X} \rightarrow \mathbb{R}$ be a measurable function, such that $\mathbb{E}_\pi f = 0$. If $\|f\|_{\pi, \psi_\beta} < \infty$ for some $\beta > 0$, then for all $t \geq 0$,*

$$\begin{aligned} & \mathbb{P}_\nu(|f(X_0) + \dots + f(X_{n-1})| \geq t) \\ & \leq K \exp\left(-\frac{t^2}{Kn\delta\pi(C)\mathbb{E}_\nu S(f)^2}\right) + K \exp\left(-\frac{t}{K\|f\|_{\pi, \psi_\beta}\|\tau + 1\|_{\nu, \psi_\alpha}^3}\right) \\ & + K \exp\left(\frac{t^\gamma}{K(\|f\|_{\pi, \psi_\beta}\|\tau + 1\|_{\nu, \psi_\alpha})^\gamma \log n}\right) \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}_\pi(|f(X_0) + \dots + f(X_{n-1})| \geq t) \\ & \leq K \exp\left(-\frac{t^2}{Kn\delta\pi(C)\mathbb{E}_\nu S(f)^2}\right) + K \exp\left(-\frac{t}{K\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha}^3}\right) \\ & + K \exp\left(\frac{t^\gamma}{K(\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha})^\gamma \log(\|\tau+1\|_{\nu,\psi_\alpha})}\right) + K \exp\left(\frac{t^\gamma}{K(\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha})^\gamma \log n}\right), \end{aligned}$$

where $\gamma = \frac{\alpha\beta}{\alpha+\beta}$ and K depends only on α, β and m in the formula (1).

Remarks

1. The proof of the above theorem is similar to those presented in [1, 2], therefore we will present only a sketch.
2. When $m = 1$, $\delta\pi(C)\mathbb{E}_\nu S(f)^2$ is the variance of the limiting Gaussian distribution for the additive functional.
3. If one does not insist on having the limiting variance in the case $m = 1$ as the subgaussian coefficient and instead replaces it by $\mathbb{E}_\nu S(f)^2$, one can get rid of the second summand on the right hand sides of the estimates (i.e. the summand containing $\|\tau+1\|_{\nu,\psi_\alpha}^3$).
4. One can also obtain similar results for suprema of empirical processes of a Markov chain (or equivalently for additive functionals with values in a Banach space). The difference is that one obtains then bounds on deviation above expectation and not from zero. A proof is almost the same, it simply requires a suitable generalization of an inequality for real valued summands, relying on the celebrated Talagrand's inequality and an additional argument to take care of the expectation. Since our goal is rather to illustrate the consequences of results from Section 2, than to provide the most general inequalities, we do not state the details and refer the reader to [1, 2] for the special case of geometrically ergodic Markov chains. For the same reason we will not try to evaluate constants in the inequalities.
5. In a similar way one can obtain tail estimates in the polynomial case (i.e. when the regeneration time and/or the function f are only polynomially integrable). One just needs to use the other examples that have been discussed in Section 2. The estimate of the bounded part (after truncation) comes again from Bernstein's inequality, whereas the unbounded part can be handled with the Hoffman-Joergensen inequality (or its easy modifications for functions of the form $x \mapsto x^p/(\log^\beta x)$), just as e.g. in [17].

Proof of Theorem 18. Below we will several times use known bounds for sums of independent random variables in the one-dependent case. Clearly it may be done at the cost of worsening the constants by splitting the sum into sums of odd and even terms, therefore we will just write the final result without further comments. In the proof we will use the letter K to denote constants depending on α, β . Their values may change from one occurrence to another.

Setting $N = \inf\{i: m\tau(i) + m - 1 \geq n - 1\}$ we may write

$$\begin{aligned} |f(X_0) + \dots + f(X_{n-1})| &= \sum_{i=0}^{(m\tau(0)+m-1)} |f(X_i)| + \left| \sum_{i=1}^N s_i(f) \right| + \sum_{i=n}^{m\tau(N)+m-1} |f(X_i)| \\ &=: I + II + III, \end{aligned}$$

where each of the sums on the right hand side may be interpreted as empty.

The first and last terms can be taken care of by Chebyshev's inequalities corresponding to proper Orlicz norms, using estimates of Corollaries 6 and 14 (note that $\mathbb{P}(III \geq t) \leq \mathbb{P}(I \geq t) + n\mathbb{P}(|s_1(f)| \geq t)$).

We will consider only the case of the chain started from ν . The stationary case is similar, simply to bound I we use the estimates of Orlicz norms for the chain started from π given in Theorem 9 (together with the remark following it to get better dependence on $\|\tau + 1\|_{\nu, \psi_\alpha}$).

By Corollary 6 and examples provided in Section 2, $\|s_i(f)\|_\varphi = \|S(f)\|_{\nu, \psi_\gamma} < K\|\tau + 1\|_{\nu, \psi_\alpha}\|f\|_{\pi, \psi_\beta}$. Thus

$$\mathbb{P}(I \geq t) + \mathbb{P}(III \geq t) \leq 2n \exp\left(\left(-\frac{t}{K\|\tau + 1\|_{\nu, \psi_\alpha}\|f\|_{\pi, \psi_\beta}}\right)^\gamma\right). \quad (18)$$

The second term can be split into $II_1 + II_2$, where

$$\begin{aligned} II_1 &= \left| \sum_{i=1}^N (s_i(f) \mathbf{1}_{\{|s_i(f)| \leq a\}} - \mathbb{E}s_i(f) \mathbf{1}_{\{|s_i(f)| \leq a\}}) \right|, \\ II_2 &= \left| \sum_{i=1}^N (s_i(f) \mathbf{1}_{\{|s_i(f)| > a\}} - \mathbb{E}s_i(f) \mathbf{1}_{\{|s_i(f)| > a\}}) \right|. \end{aligned}$$

Setting $a = K_{\alpha, \beta} \max_{i \leq n} \|s_i(f)\|_{\psi_\gamma} \log^{1/\gamma} n \leq K_{\alpha, \beta} m \|f\|_{\pi, \psi_\beta} \|\tau + 1\|_{\psi_\alpha} \log^{1/\gamma} n$, we can proceed as in [2] to get

$$\mathbb{P}(II_2 \geq t) \leq 2 \exp\left(\left(-\frac{t}{K_{\alpha, \beta} a}\right)^\gamma\right). \quad (19)$$

It remains to bound the term I_1 . Introduce the variables $T_i = \tau(i) - \tau(i-1)$, $i \geq 1$ and note that $\mathbb{E}T_i = \delta^{-1}\pi(C)^{-1}$. For $4nm^{-1}\pi(C)\delta \geq 2$, we have

$$\begin{aligned} \mathbb{P}(N \geq 4nm^{-1}\pi(C)\delta) &\leq \mathbb{P}\left(\sum_{i=1}^{\lfloor 4nm^{-1}\pi(C)\delta \rfloor} T_i \leq n/m\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{\lfloor 4nm^{-1}\pi(C)\delta \rfloor} (T_i - \mathbb{E}T_i) \leq n/m - 2nm^{-1}\pi(C)\delta \mathbb{E}T_i\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{\lfloor 4nm^{-1}\pi(C)\delta \rfloor} (T_i - \mathbb{E}T_i) \leq -n/m\right) \\ &\leq k(n/m), \end{aligned}$$

where $k(t) = K_\alpha \exp(-K_\alpha^{-1} \min(t^2 m/n \|\tau + 1\|_{\nu, \psi_\alpha}^2, (t/\|\tau + 1\|_{\nu, \psi_\alpha})^\alpha))$. The bound follows for $\alpha = 1$ from Bernstein's ψ_1 inequality and for $\alpha < 1$ from results in [19] (as shown in [3]).

Note that if $4nm^{-1}\pi(C)\delta < 2$, then

$$\begin{aligned} k(n/m) &\geq \exp(-K_\alpha nm^{-1} \|\tau + 1\|_{\nu, \psi_\alpha}^{-2}) \geq \exp(-K_\alpha nm^{-1} (\mathbb{E}\tau + 1)^{-2}) \\ &= \exp(-K_\alpha nm^{-1} \delta \pi(C)) \geq \exp(-K_\alpha/2). \end{aligned}$$

Thus the above tail estimate for N remains true (after adjusting the constant K_α).

Thus by Bernstein's bounds on suprema of partial sums of a sequence of independent random variables we get

$$\begin{aligned} \mathbb{P}(II_1 \geq t) &\leq \mathbb{P}(II_1 \geq t \ \& \ N \leq 4nm^{-1}\pi(C)\delta) + k(n/m) \\ &\leq K \exp\left(-\frac{1}{K} \min\left(\frac{t^2}{nm^{-1}\pi(C)\delta \mathbb{E}_\nu S(f)^2}, \frac{t}{\|f\|_{\pi, \psi_\beta} \|\tau\|_{\psi_\alpha}}\right)\right) + k(n/m), \end{aligned} \quad (20)$$

On the other hand by $N \leq n/m$, the same inequalities we used to derive the function k and Levy type inequalities (like in [1]), we obtain

$$\mathbb{P}(II \geq t) \leq h(t), \quad (21)$$

where $h(t) = K_\gamma \exp(-K_\gamma^{-1} \min(t^2 m/n (\|f\|_{\pi, \psi_\beta} \|\tau + 1\|_{\nu, \psi_\alpha})^2, (t/\|f\|_{\pi, \psi_\beta} \|\tau + 1\|_{\nu, \psi_\alpha})^\gamma))$

Now if $k(n/m) \geq K \exp(-(t/K \|f\|_{\pi, \psi_\beta} \|\tau + 1\|_{\nu, \psi_\alpha})^\alpha)$, then

$$K \exp\left(-\left(\frac{t}{K \|f\|_{\pi, \psi_\beta} \|\tau + 1\|_{\nu, \psi_\alpha}}\right)^\alpha\right) \leq k(n/m) \leq K \exp\left(-\left(\frac{n}{Km \|\tau + 1\|_{\nu, \psi_\alpha}^2}\right)^\alpha\right)$$

and so

$$\frac{t^2 m}{n (\|f\|_{\pi, \psi_\beta} \|\tau + 1\|_{\nu, \psi_\alpha})^2} \geq \frac{t}{\|\tau + 1\|_{\nu, \psi_\alpha}^3 \|f\|_{\pi, \psi_\beta}},$$

which ends the proof by (21). \square

Appendix. Some generalities on Orlicz Young functions and Orlicz spaces

All the lemmas presented below are standard facts from the theory of Orlicz spaces, we present them here for the reader's convenience.

Lemma 19. *If φ is a Young function then $X \in L_\varphi$ if and only if $\mathbb{E}|XY| < \infty$ for all Y such that $\mathbb{E}\varphi^*(Y) \leq 1$. Moreover the norm $\|X\| = \sup\{\mathbb{E}XY : \mathbb{E}\varphi^*(Y) \leq 1\}$ is equivalent to $\|X\|_\varphi$.*

The next lemma is a modification of Lemma 5.4. in [27]. In the original formulation it concerns the notion of equivalence of functions (and not asymptotic equivalence relevant in our probabilistic setting). One can however easily see that the proof from [27] yields the version stated below.

Lemma 20. Consider two increasing continuous functions $F, G: [0, \infty) \rightarrow [0, \infty)$ with $F(0) = G(0) = 0$, $F(\infty) = G(\infty) = \infty$. The following conditions are equivalent

(i) $F \circ G^{-1}$ is equivalent to a Young function.

(ii) There exist positive constants C, x_0 such that

$$F \circ G^{-1}(sx) \geq C^{-1} s F \circ G^{-1}(x)$$

for all $s \geq 1$ and $x \geq x_0$.

(iii) There exist positive constants C, x_0 such that

$$\frac{F(sx)}{F(x)} \geq C^{-1} \frac{G(sx)}{G(x)}$$

for all $s \geq 1$, $x \geq x_0$.

Lemma 21. For any Young function ψ such that $\lim_{x \rightarrow \infty} \psi(x)/x = \infty$ and any $x \geq 0$,

$$x \leq (\psi^*)^{-1}(x) \psi^{-1}(x) \leq 2x.$$

Moreover the right hand side inequality holds for any strictly increasing function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$, $\psi(\infty) = \infty$, $\lim_{x \rightarrow \infty} \psi(x)/x = \infty$.

Lemma 22. Let φ and ψ be two Young functions. Assume that $\lim_{x \rightarrow \infty} \varphi(x)/x = \infty$. If $\varphi^{-1} \circ \psi$ is equivalent to a Young function, then so is $(\psi^*)^{-1} \circ \varphi^*$.

Proof. It is easy to see that under the assumptions of the lemma we also have $\lim_{x \rightarrow \infty} \psi(x)/x = \infty$ and thus $\varphi^*(x)$, $\psi^*(x)$ are finite for all $x \geq 0$. Applying Lemma 20 with $F = \varphi^{-1}$, $G = \psi^{-1}$, we get that

$$\frac{\varphi^{-1}(sx)}{\psi^{-1}(sx)} \geq C^{-1} \frac{\varphi^{-1}(x)}{\psi^{-1}(x)}$$

for some $C > 0$, all $s \geq 1$ and x enough. By Lemma 21 we obtain for x large enough,

$$\frac{(\psi^*)^{-1}(sx)}{(\varphi^*)^{-1}(sx)} \geq (4C)^{-1} \frac{(\psi^*)^{-1}(x)}{(\varphi^*)^{-1}(x)},$$

which by another application of Lemma 20 ends the proof. □

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